ERROR PROPAGATION IN ANGLE MEASUREMENTS

SOURCES OF ERRORS

1. Reading the circle
   personal value

2. Pointing on the target
   personal value
   dependent on instrument

3. Target setup
   reduced by increasing sight distance

4. Instrument setup
   reduced by increasing sight distance

5. Instrument misleveling
   largest affect when altitude of target high

READING ERROR
REPETITION METHOD

WHEN READINGS ARE TAKEN:

1) read circle when zeroing circles

2) read circle after final pointing

FOR ANGLE TURNED *n* TIMES USinG BOTH FACES:

$$AVE.\ \angle = \frac{FINAL\ \ READING\ - \ INITIAL\ \ READING}{n}$$

APPLYING ERROR IN SUM:

$$\sigma_{\alpha_r} = \sqrt{\left(\frac{\sigma_f}{n}\right)^2 + \left(\frac{\sigma_0}{n}\right)^2}$$

READING ERROR
REPETITION METHOD

If $\sigma_f = \sigma_0 = \sigma_r$, then

$$\sigma_{\alpha_r} = \frac{\sigma_r \sqrt{2}}{n}$$

EXAMPLE

Assume $\sigma_r = \pm 1.5''$ and angle is measured 6 times using the repetition method, what is the expected error in the angle due to the reading error:

$$\sigma_{\alpha_r} = \pm \frac{1.5\sqrt{2}}{6} = \pm 0.4''$$

READING ERROR
DIRECTIONAL METHOD

WHEN READINGS ARE TAKEN:

1) read circle for each pointing

2) angle an indirect measurement,

$$\angle = \text{READING}_{FS} - \text{READING}_{BS}$$

For angle turned *n* times using both faces:

$$AVE \ \angle = \frac{(R_{FS1} - R_{BS1}) + (R_{FS2} - R_{BS2}) + \cdots + (R_{FSn} - R_{BSn})}{n}$$

where R_{xxx} is a circle reading for the xxx sight.

APPLY ERROR IN SUM:

$$\sigma_{\alpha_r} = \frac{1}{n}\sqrt{\sigma^2_{FS1_r} + \sigma^2_{BS1_r} + \sigma^2_{FS2_r} + \sigma^2_{BS2_r} + \cdots + \sigma^2_{FSn_r} + \sigma^2_{BSn_r}}$$

If $\sigma_r = \sigma_{FS1_r} = \Sigma_{BS1_r} = \sigma_{FS2_r} = \Sigma_{BS2_r} = \cdots$

Then

$$\sigma_{\alpha_r} = \frac{\sqrt{n \times 2 \times \sigma_r^2}}{n} = \frac{\sigma_r \sqrt{2}}{\sqrt{n}}$$

READING ERROR
DIRECTIONAL METHOD

EXAMPLE:

Assume $\sigma_r = \pm 1.5''$ and angle is measured 6 times using the directional method, what is the expected error in the angle due to the reading error:

$$\sigma_{\alpha_r} = \pm \frac{1.5\sqrt{2}}{\sqrt{6}} = \pm 0.87''$$

ERRORS IN ANGLES DUE TO POINTING

Factors that affect pointing error size:

a) optical qualities of telescope

b) target size

c) personal ability of observer to center on target

d) weather conditions (fog, heat waves, etc.)

ANALYSIS OF POINTING ERRORS

Every angle involves 2 pointings, thus for *n* angle measurements.

$$\sigma_{\alpha_p} = \frac{\sqrt{\sigma^2_{p_{BS_1}} + \sigma^2_{p_{FS_1}} + \sigma^2_{p_{BS_2}} + \sigma^2_{p_{FS_2}} + \cdots + \sigma^2_{p_{BS_n}} + \sigma^2_{p_{FS_n}}}}{n}$$

If $\sigma_{p_{BS_1}} = \sigma_{p_{FS_1}} = \sigma_{p_{BS_2}} = \sigma_{p_{FS_2}} = \cdots = \sigma_{p_{BS_n}} = \sigma_{p_{FS_n}} = \sigma_p$

Then by applying the **error in the sum**,

$$\sigma_{\alpha_p} = \frac{\sigma_p \sqrt{2n}}{n}$$

$$= \sigma_p \frac{\sqrt{2}}{\sqrt{n}}$$

EXAMPLE

An angle measured 6 times by an observer whose ability to point on a well-defined target is estimated to be ±1.8". What is the expected error in the angle due to pointing?

$$\sigma_{\alpha_p} = 1.8^{\prime\prime}\frac{\sqrt{2}}{\sqrt{6}}$$

$$= \pm 1.0^{\prime\prime}$$

ESTIMATED POINTING AND READING ERROR WITH TOTAL STATIONS

DIN 18723 STANDARD:

Based on a single direction measured with both faces of the instrument.

$$\sigma_{pr} = \sigma_{DIN}\sqrt{2}$$

Then for *n* measurements of an angle:

$$\sigma_{\alpha_{pr}} = \frac{2\,\sigma_{DIN}}{\sqrt{n}}$$

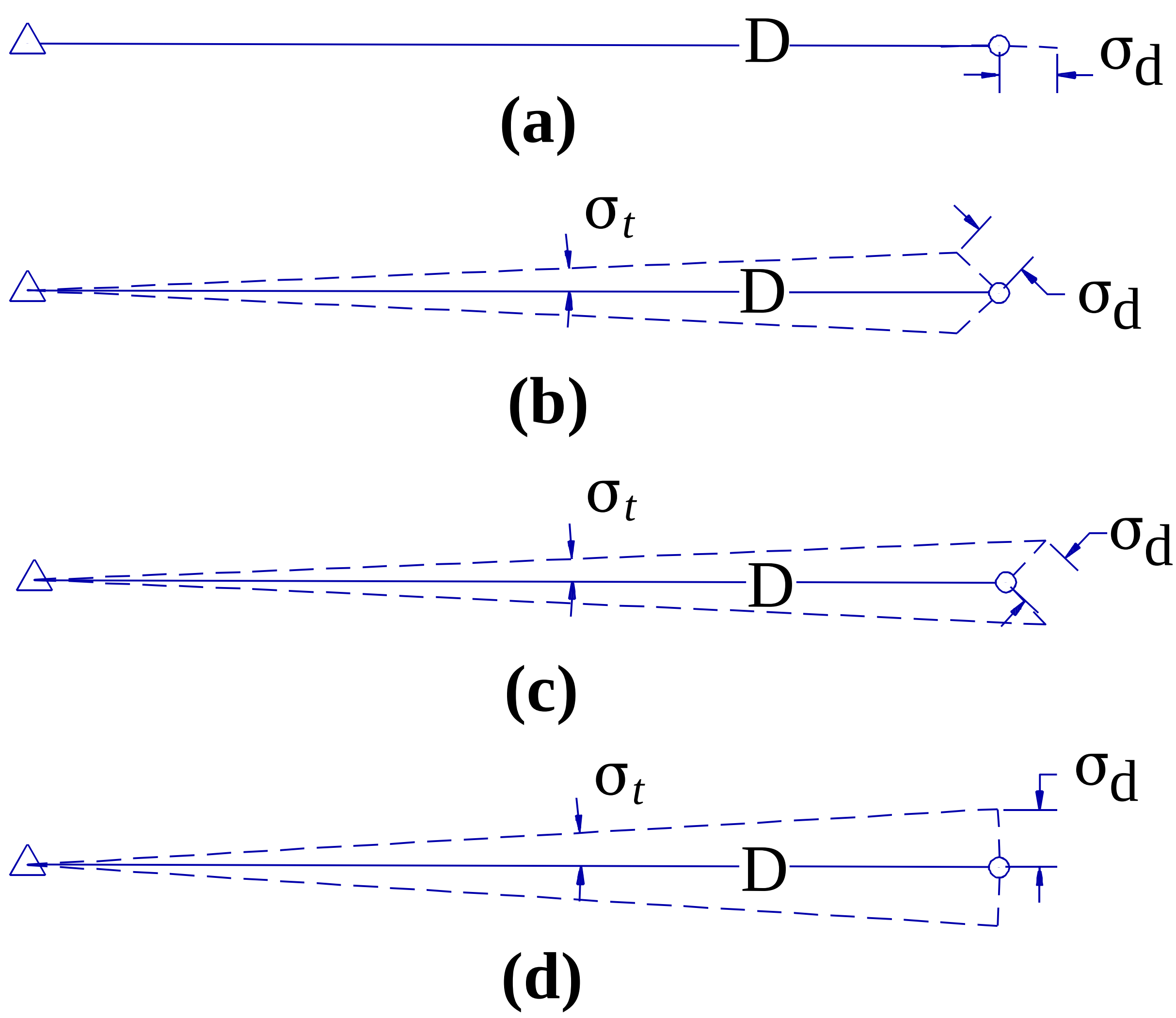
EXAMPLE

An angle measured 6 times by an observer with a total station having a DIN18273 value of ±5". What is the estimated error in the angle due to pointing and reading?

$$\sigma_{\alpha_{pr}} = \pm\frac{2 \times 5''}{\sqrt{6}}$$

$$= \pm 4.1''$$

ERRORS IN ANGLES DUE TO TARGET MISCENTERING



NOTE: This error is systematic for an individual setup, but will appear random over multiple setups.

ERROR IN A SINGLE DIRECTION

$$\sigma_t = \frac{\sigma_d}{D}$$

where σ_t is in radians.

ERROR IN AN ANGLE

$$\sigma_{\alpha_t} = \sqrt{\left(\frac{\sigma_{d_1}}{D_1}\right)^2 + \left(\frac{\sigma_{d_2}}{D_2}\right)^2}$$

IF $\sigma_{d_1} = \sigma_{d_2} = \sigma_t$, then:

$$\sigma_{\alpha_t} = \sqrt{\left(\frac{\sigma_t}{D_1}\right)^2 + \left(\frac{\sigma_t}{D_2}\right)^2}$$

$$\sigma_{\alpha_t} = \sigma_t \sqrt{\left(\frac{1}{D_1}\right)^2 + \left(\frac{1}{D_2}\right)^2}$$

ERROR IN AN ANGLE

$$\sigma_{\alpha_t} = \sigma_t \frac{\sqrt{D_1^2 + D_2^2}}{D_1 D_2}$$

where σ_{α_t} is in radians

NOTE: Since same error occurs on every pointing, it cannot be reduced in size by multiple pointings.

EXAMPLE

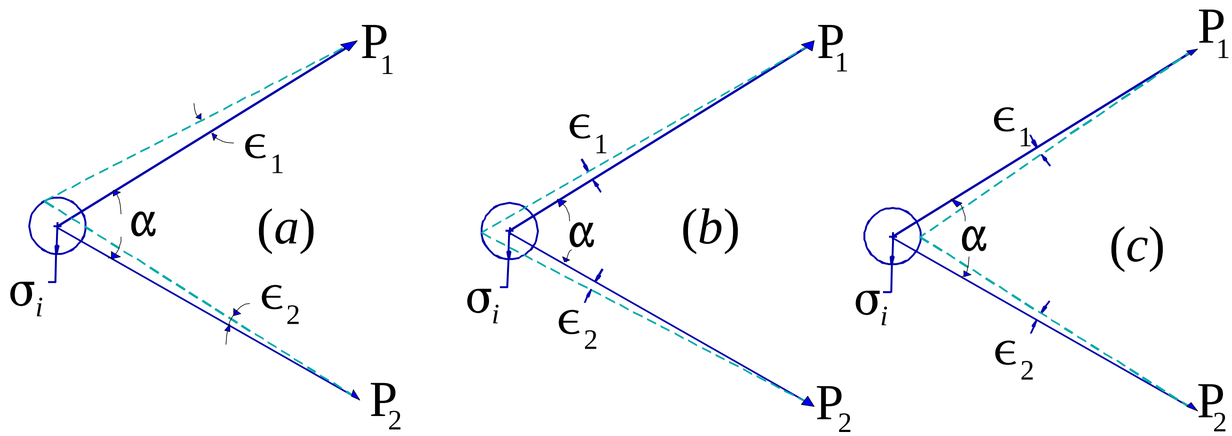
Hand held targets are centered over a station to within ±0.02 ft. What is the error in the angle due to target centering if the backsight distance is 250 ft and the foresight distance is 150 ft?

$$\sigma_{\alpha_t} = 0.02 \; \frac{\sqrt{250^2 \; + \; 150^2}}{250 \times 150} \times \rho$$

$$= \pm 32''$$

where ρ = 206,264.8"/radian

INSTRUMENT MISCENTERING



Facts:

Error in every direction due to instrument miscentering.

At 2 positions one of which is shown in (*a*), the errors cancel.

Although systematic for a particular setup, they appear random with mulitple setups and multiple stations.

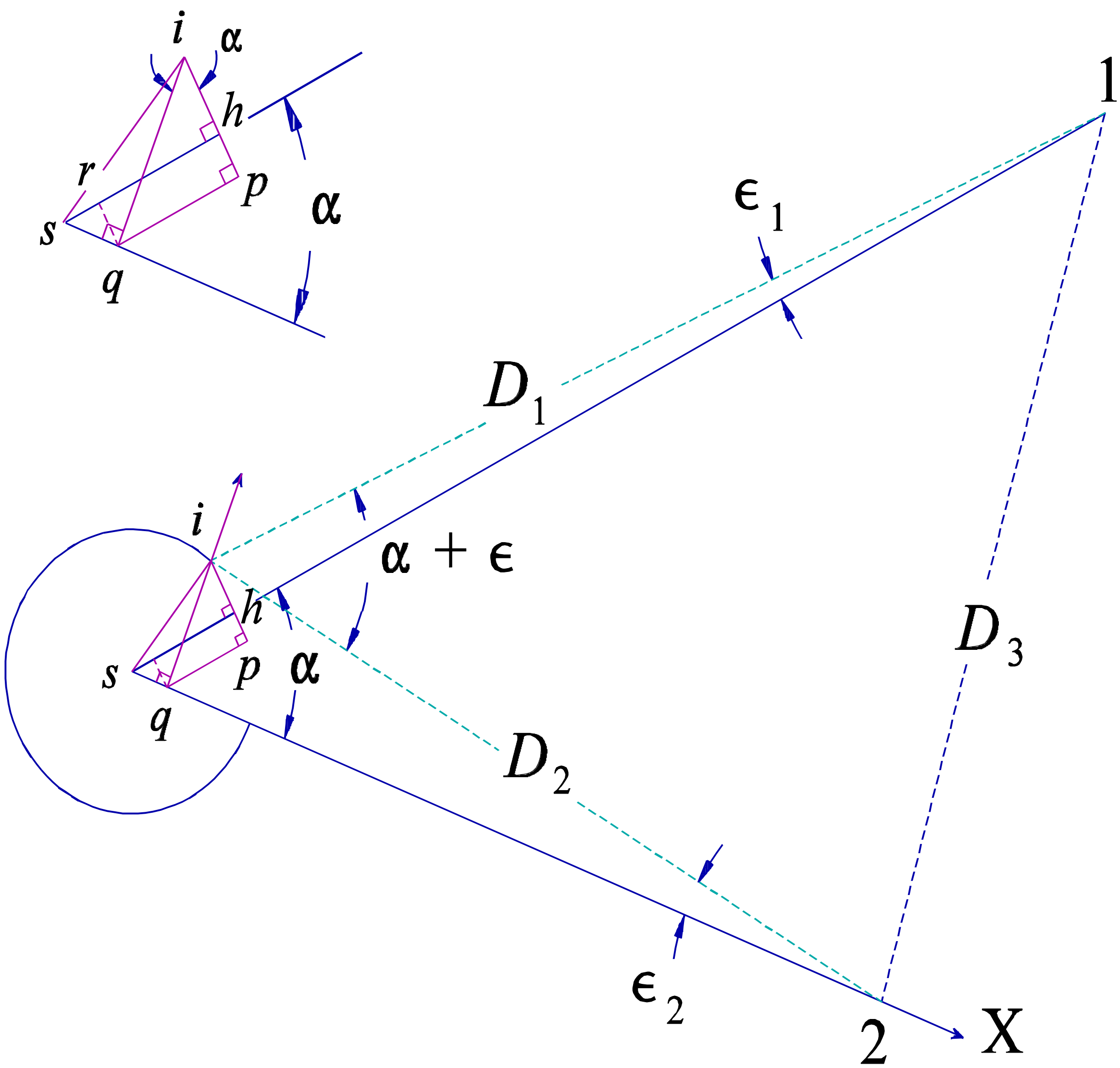
ERROR IN AN INDIVIDUAL POINTING

$$\alpha = (P_2 + \epsilon_2) - (P_1 + \epsilon_1)$$

$$= (P_2 - P_1) + (\epsilon_2 - \epsilon_1)$$

$$= (P_2 - P_1) + \epsilon$$

ANALYSIS OF ERROR CONTRIBUTION



FROM SKETCH

$$ih = ip - qr$$

$$ih = iq \cos(\alpha) - sq \sin(\alpha)$$

Let $sq = x$ and $iq = y$, then

$$ih = y \cos(\alpha) - x \sin(\alpha)$$

NOW

$$\epsilon_1 = \frac{ih}{D_1} = \frac{y \cos(\alpha) - x \sin(\alpha)}{D_1}$$

$$\epsilon_2 = y/D_2$$

$$\boxed{\epsilon = \frac{y}{D_2} - \frac{y \cos(\alpha) - x \sin(\alpha)}{D_1}}$$

OR

$$\boxed{\epsilon = \frac{D_1 y + D_2 x \sin(\alpha) - D_2 y \cos(\alpha)}{D_1 D_2}}$$

ANALYSIS OF ERROR DUE TO INSTRUMENT MISCENTERING

Applying SLOPOV the parital derivative *wrt x* is

$$\frac{\partial \epsilon}{\partial x} = \frac{D_2 \sin(\alpha)}{D_1 D_2}$$

and *wrt y* is

$$\frac{\partial \epsilon}{\partial y} = \frac{D_1 - D_2 \cos(\alpha)}{D_1 D_2}$$

ANALYSIS OF ERROR DUE TO INSTRUMENT MISCENTERING

Since an error in x is as likely as the error in y, and since this represents a bivariate distribution:

$$\sigma_x = \sigma_y = \frac{\sigma_i}{\sqrt{2}}$$

Applying slopov, the error in the angle is:

$$\sigma^2_{\alpha_i} = \frac{D_1^2 + D_2^2(\cos^2(\alpha) + \sin^2(\alpha)) - 2D_1D_2\cos(\alpha)}{D_1^2D_2^2}\,\frac{\sigma_i^2}{2}$$

ANALYSIS OF ERROR DUE TO INSTRUMENT MISCENTERING

Rearranging, this can be simplified to:

$$\boxed{\sigma_{\alpha_i} = \pm \frac{D_3}{D_1 D_2 \sqrt{2}} \sigma_i}$$

Where σ_{α_i} is in radians

Mulitply by ρ = 206,264.8"/radian to get value in seconds.

EXAMPLE

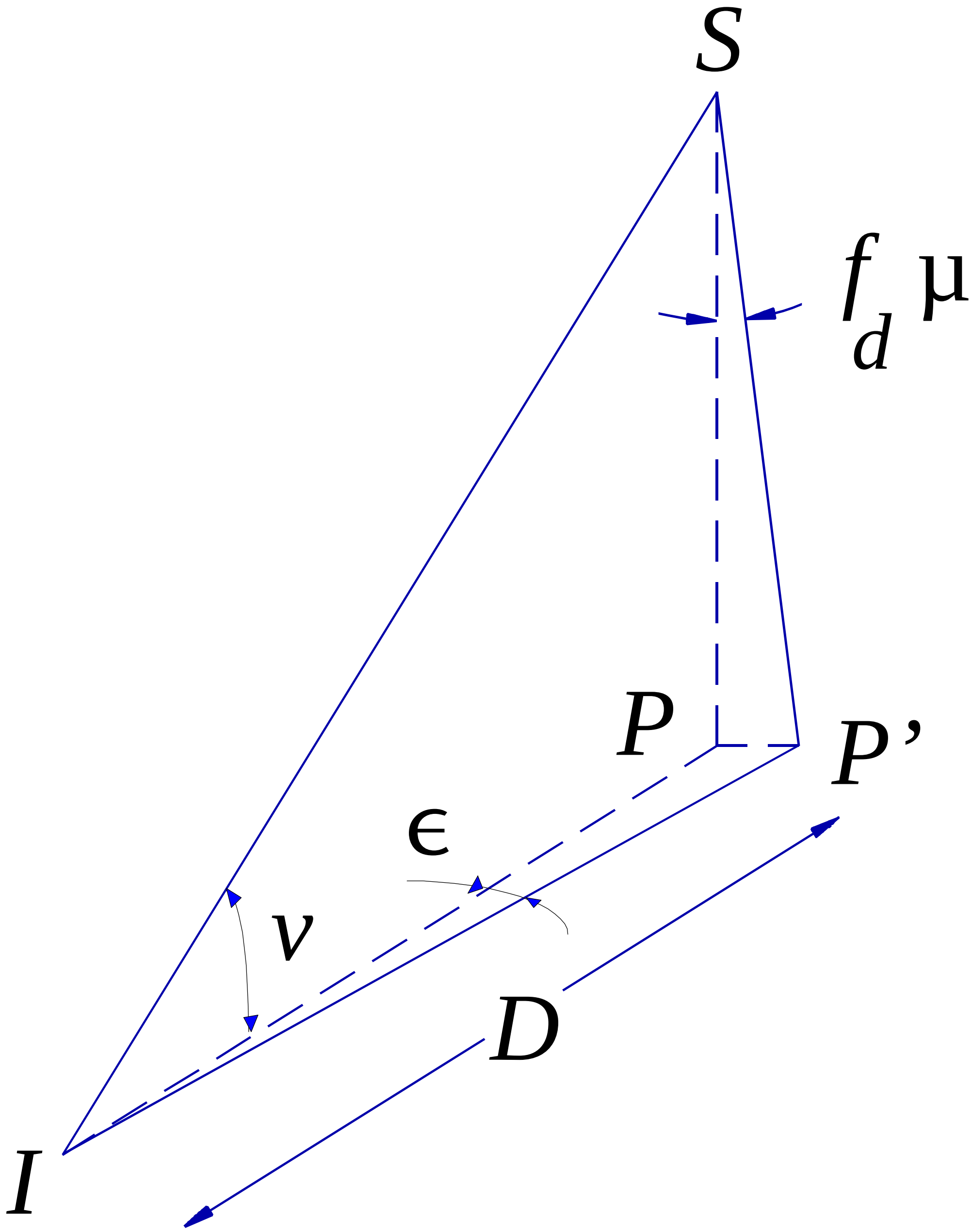
What is the error in a 50° angle due to instrument miscentering, if the setup is within ±0.005 ft and the sight distances are 150 and 250 ft?

$$D_3 = \sqrt{150^2 + 250^2 - 2\,(150)\,(250)\,Cos(50)} = 191.81$$

So

$$\sigma_{\alpha_i} = 0.005\,\frac{191.81}{(150)\,(250)\,\sqrt{2}}\,\rho = \pm 3.7''$$

EFFECTS OF MISLEVELING



EFFECTS OF MISLEVELING

DERIVATION:

$$SP = D \tan(v)$$

$$PP' = D\epsilon$$

where D is the sighting distance and the angular error ϵ is in radians

$$PP' = f_d \mu \, (SP)$$

where f_d is the fractional division of the bubble and μ is the sensitivity of the bubble

But $$PP' = f_d \mu D \tan(v)$$

EFFECTS OF MISLEVELING

So the error in the direction due to misleveling is

$$\epsilon = f_d\, \mu \tan(v)$$

For an angle measured n times

$$\sigma_{\alpha_l} = \pm \frac{\sqrt{(f_d\, \mu \tan(v_b))^2 + (f_d\, \mu \tan(v_f))^2}}{\sqrt{n}}$$

EXAMPLE

How much error in the horizontal angle can be expected in a sun shot, if the instrument has a bubble with a sensitivity of 30"/div and is leveled to within 0.5 divisions. Assume a backsight zenith angle of 91°30'45" and a foresight zenith angle to the sun of 55°15'30"? (Assume angle turned 3DR.)

$$\sigma_{\alpha_l} = \sqrt{[0.5\,(30)\tan(1°30'45'')]^2 + [0.5\,(30)\tan(34°44'30'')]^2}/\sqrt{6}$$

$$= \sqrt{0.4^2 + 10.4^2}/\sqrt{6}$$

$$= \pm 4.2''$$

ESTIMATED ERROR IN A HORIZONTAL ANGLE

Assume a rectangular parcel, with sides of 251.75 and 347.05 ft (2 acre parcel). Also assume that

$\sigma_{DIN} = \pm 5''$
$n = 2$

Method: Directional
$\sigma_i = \pm 0.005$ ft
$\sigma_t = \pm 0.01$ ft

What is the expected error in an angle? What is the expected angular misclosure?

Solution: Solve each error component separately, and then sum these using error in sum.

Pointing and reading contribution:

$$\sigma_{\alpha_{pr}} = \frac{2 \times 5''}{\sqrt{2}}$$

$$= \pm 7.1''$$

Target miscentering:

$$\sigma_{\alpha_t} = 0.01 \frac{\sqrt{251.75^2 + 347.05^2}}{(251.75)(347.05)} \rho$$

$$= \pm 10.1''$$

Instrument miscentering:

$$D_3 = \sqrt{251.75^2 + 347.05^2} = 428.744\,ft$$

$$\sigma_{\alpha_i} = \frac{0.005}{\sqrt{2}} \cdot \frac{428.744}{(251.75)(347.05)}\,\rho$$

$$= \pm 3.6^{\prime\prime}$$

Error in a single measured angle:

$$\sigma_\alpha = \sqrt{7.1^2 + 10.1^2 + 3.6^2}$$

$$= \pm 12.9''$$

EXPECTED ERROR
IN SUM OF ANGLES:

$$\sigma_{\sum \measuredangle} = \pm 13.4\sqrt{4} = \pm 26.8''$$

This is the value that you would expect a field crew to close within 68% of the time.

If you were checking closure then use a 95% confidence level, or thus a $t_{\alpha,\, n\text{-}1}$ multiplier.

Since each angle measured only twice, n - 1 = 1 and

$$t_{0.025,\, 1} = 12.705$$

So

$$\sigma_{\sum \measuredangle_{95\%}} = 12.705 \times \frac{26.8}{\sqrt{2}} = \pm 241''$$