ERROR PROPAGATION OF COORDINATE DETERMINATION

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Introduction

In the following paper and accompanying worksheet I will go over the random sources of error that affect the measurements of a modern total station and how they influence the derived coordinates of a surveyed point. The calculated precisions assume that the systematic errors have been accounted for and properly reduced with calibration and field procedures.

Sources of Error	
Random Error	**Systematic Error**
• Pointing & Reading (σd) – *Provided by manufacturer* • EDM Constant (σE_c)) – *Provided by manufacturer* • EDM Scale (σE_s)) – *Provided by manufacturer* • Centering error @ Instrument over the point (σI) - *Estimated* • Centering error @ Target over the point (σT) - *Estimated* • Plumbness of Prism pole error (σP) - *Calculated* • Centering error of the reflector optical center vs center of prism holder (σR) – *Provided by manufacturer* • Measurement of height of instrument (σV_i) - *Estimated* • Measurement of height of target (σV_t) - *Estimated*	• Horizontal Collimation • Tilting Axis • Vertical Collimation • Compensator index • Circle Eccentricity • Circle Graduation • Collimation of EDM *These can be reduced by F1 + F2 observations and regular calibration • Environmental conditions • Un-even heating of the instrument / calibration at a different temperature • Vibrations/tripod instability • Prism pole out of level • EDM out of calibration *These can be reduced equipment calibration & field procedures.

Pointing & Reading error (σd)

Pointing and reading error is specified by the manufacturer in +/- arc seconds (σd) in ISO (17123-3) or DIN (18723) standards. This is the angular standard deviation of a **single direction** of a f1+f2 measurement. To get the standard deviation of an **angle**, multiply the supplied standard deviation by √2, because there are two pointings in a measured angle. One at the back sight and one at the foresight.

$$\sigma A_g = \sigma d * \sqrt{n}$$

$$Ex.\ \sigma A_g = 3'' * \sqrt{2} \text{ (for a 3'' instrument)}$$

$$\sigma A_g = 4.24''$$

To get the standard deviation of the angle observed over multiple sets you must divide the standard deviation by √n where n is the number of sets.

$$\sigma A_g = {}_{+/-}\,({\sigma d} * \sqrt{2}) / \sqrt{n}$$

Ex. 4.24" / √3 (for 3 rounds of observations of an angle)

$$\sigma A_g = 2.45''$$

To get the standard deviation of the angle observed with a single face (f1 only), substitute n = 0.5 into the same equation as you are taking 0.5 sets of observations.

$$\sigma A_g = (\sigma d * \sqrt{2}) / \sqrt{n}$$

Ex. 4.24" / √0.5 (for a single face observation of an angle and a 3" instrument)

$$\sigma A_g = 6''$$

EDM Error (σEc + σEs)

The EDM constant error is specified by the manufacturer (ISO 17123-4) with two components, the constant, in +/- meters or mm (σE_c) and a scale component (σE_s) that's in PPM. This quoted accuracy is based off a single measured distance, such as in tracking mode. However, it is common for the instrument to take 3-5 (depending on settings) distance observations and use the mean of those measurements.

The σEc + σEs may be different for IR (infrared) vs RL (reflectorless) measurements, and it should be noted that RL EDM measurements are only at their quoted accuracy when shooting a target perpendicular to the LOS (line of site) with good reflectivity and a flat surface, free of obstructions. The accuracy degrades significantly past 30° and should be avoided. To calculate the combined effect these two components have on the measured distance (σSD) we must use the following equation where n_D is the number of distance measurements taken by the instrument.

$$\sigma SD = +/- (\sigma E_c + (\sigma E_s * SD)) / \sqrt{n_D}$$

*Ex. σSD = (0.002 + ((2 / 1000000) * 500) / √5 (for a 2mm +/- 2 PPM instrument @ 500m measured with 5 distance measurements taken)*

$$\sigma SD = 0.0013m$$

*Ex. σSD = (0.002 + ((2 / 1000000) * 500) / √1 (for a 2mm +/- 2 PPM instrument @ 500m measured in tracking mode)*

$$\sigma SD = 0.0030m$$

Centering error (horizontally) of the reflector glass over the point (σR)

Different reflectors are made to different quality standards. However, all reflectors contain some quantity of centering error. That is the amount the optical center of the prism may vary when compared to the vertical axis of the prism holder. These values can be found on most manufacturers' websites. They generally range from 0.0003m to 0.002m. This is not to be confused with beam deviation. When a reflector is quoted as having an X " of beam deviation, that value affects the maximum range of the EDM, not so much the precision of the distance measurement. The more the EDM beam diverges, the more the range is reduced.

Centering error (horizontally) over a point ($\sigma I + \sigma T + \sigma P$)

The ability to perfectly center the instrument and target over the point horizontally affects the measured slope distance (SD), derived horizontal distance (HD), the measured azimuth (AZ) and subsequently the derived horizontal position of the point (N & E). These can be estimated based on how well the points center can be defined, the condition of the tribrach/prism pole, whether a bipod was used and the height of instrument/target. This should not be confused as the centering error provided by the manufacturer of a reflector based on the precision in which the reflector was constructed (σR).

An old, beat-up tribrach that hasn't been recently adjusted, on a tripod will have a higher centering error than a newer tribrach in good condition. And a prism pole at a 2m HT (height of target) without a bipod is going to have a larger standard deviation due to plumbness of pole (σP) versus a mini prism with a HT of 0.100m. This value can be approximated with a function based on the bubble sensitivity of a pole (P_s) and the HT. The bubble sensitivity is defined as the angular deviation of the prism pole with respect to the vertical axis that is present when a level bubble moves 2 mm from the center. The idea is the surveyor wouldn't allow the bubble to be out of center by more than 2mm. At that distance, the edge of the bubble touches the sides of the black circle on the viewing glass (Leica). If you are unsure of the sensitively of your poles level bubble you can assume it is 20'. 8' bubbles are available for higher precision work and some topo poles have 40' bubbles.

$$\sigma P = HT * \tan(P_s)$$

Ex. $\sigma P = 2.000 * \tan(20')$ *(for a 2m topo rod)*

$$\sigma P = 0.012m$$

Ex. $0.10 * \tan(20')$ *(for a mini prism at its lowest extension)*

$$\sigma P = 0.0006m$$

As you can see there is a significant difference based on your HT, this can be greatly reduced using a bipod. If a bipod is used, I recommend reducing the affect P_s has on the derived error by 4x. This is because now the surveyor is capable of precisely holding the bubble within the middle by 0.5mm. This assumes the pole is straight and the level bubble is calibrated. The equation to be used with a bipod is as follows:

$$\sigma P = HT * \tan(P_s / 4)$$

Ex. $\sigma P = 2.000 * \tan(20' / 4)$ *(for a 2m topo rod)*

$$\sigma P = 0.003$$

This equation assumes the pole tip is precisely centered on the point, which is easier done with forced centering points, a mag nail or capped post for example.

If a tribrach and prism are used, then σP should be replaced with σT, which is the standard deviation of the target centering over a point and is due to imperfect levelling, imperfections in perpendicularity of the optical plummet and not being able to precisely make out the center of the point while viewing it through the optical plummet. This is more difficult to calculate based on the variables of the equipment condition and viewability of the point. I recommend estimating a value of 0.003m – 0.005m if the point has a well-defined center and the tribrach is in good condition. The perpendicularity of the optical plummet can be checked with tribraches that rotate by centering the point, rotating the tribrach, and checking to see if the point is still centered. This is not possible with fixed tribraches. To check these, you can center the tribrach, then install an instrument in good repair onto it and use the instruments eyepiece to verify it agrees with the tribrach. The centering error at the instrument (σI) has the same contributors as that of σT, however since the instrument has a much more sensitive level bubble and generally the tribrach is stored in better conditions, I tend to use a smaller value here. Somewhere around 0.002m if your setup point has a well-defined center.

Measurement of height of instrument & target ($\sigma V_i + \sigma V_t$)

When we measure the HI or HT there are random errors associated with the measurements of those heights. These values can be estimated as there is no accurate way to calculate them. The error associated with measuring the HI (σV_i) is generally smaller than that of measuring the HT (σV_t) as the most instruments tend to have a more well-defined vertical center compared to most reflector back plates. This is especially true when using the bottom mark that's present in newer total stations. This rule is reversed when using prism poles that have snap in vertical graduations, the Leica GLS12 for example, or a mini prism with screw together extensions. I generally tend to use a value of 0.003m for σV_i and a value of 0.005m for σV_t when using a prism on a tripod and 0.001-0.003m when using a prism pole (depending on the model of pole and the condition of the tip).

Error in determining an azimuth (σAZ)

We know the azimuth to a point can be determined by a function of the azimuth at the backsight (AZ_{BS}) and the measured angle (A_g)

$$AZ = AZ_{BS} + A_g - 180$$

The standard deviation of the azimuth (σAZ) to the foresight can be determined using the law of error propagation on the function and adding in the effect of the horizontal error due to centering as a ratio over the measured distance;

$$\sigma AZ = \sqrt{}\,((\sigma AZ_{BS})^2 + (\sigma A_g)^2 + (\sigma I / HD)^2 + (\sigma P / HD)^2 + (\sigma R / HD)^2)$$

Since we are assuming the control we are coming off of are without error and are holding them as fixed (we are calculating relative accuracy), then we can assume the derived azimuth at the backsight is also without error. Therefor;

$$\sigma AZ = \sqrt{}\ (0)^2 + (\sigma A_g)^2 + (\sigma I/HD)^2 + (\sigma P/HD)^2 + (\sigma R/HD)^2$$

(Continuing on from our above example with a HT = 1.500 and using a bipod)

$$Ex.\ \sigma AZ = \sqrt{}(((3'' * \sqrt{}2) / \sqrt{}1)^2 + (0.002/499.981)^2 + (0.0022/499.981)^2 + (0.002/499.981)^2)/(\pi/648000)$$

$$Ex.\ \sigma AZ = 4.49''$$

Error in determining the horizontal distance (σHD)

The horizontal distance (HD) is the slope distance reduced with the vertical component removed. It is a function of the SD and the zenith angle (ZA)

$$HD = SD * \sin(ZA)$$

Using error propagation, we can determine the standard deviation of the horizontal distance (σHD) is equal to;

$$\sigma HD = \sqrt{(\partial HD/\partial SD)^2 * \sigma SD^2 + (\partial HD/\partial ZA)^2 * \sigma ZA^2}$$

σZA is equal to the standard deviation of a single direction (σd) because a zenith angle is not the result of two pointings, but a single pointing relative to the vertical axis of the instrument as defined by gravity at that point. Therefor;

$$(\partial HD/\partial SD) = \sin(ZA)$$

$$(\partial HD/\partial ZA) = SD * \cos(ZA)$$

$$\sigma HD = \sqrt{(\sin(ZA)^2 * (\sigma E_c + (\sigma E_s * SD))^2 + (SD * \cos(ZA))^2 * (\sigma d / \sqrt{n})^2)}$$

$$Ex.\ \sqrt{(\sin(90.5)^2 * (0.002 + ((2/1000000)*500))^2 + (500 * \cos(90.5))^2 * (3'' * (\pi/648{,}000) / \sqrt{1})^2)}$$

(for a 2mm +/- 2 PPM, 3” instrument @ a SD = 500m and ZA = 90° 30’ 00” in tracking mode)

$$\sigma HD = 0.0030m$$

The total standard deviation of the horizontal distance (σHD_t) also includes the errors from centering the instrument over the point (σI), target (σT) (or pole plumbness (σP)) and the centering error of the reflector (σR). Thus the final equation for the σHD is as follows;

$$\sigma HD_t = \sqrt{(\sigma HD^2 + \sigma I^2 + \sigma P \text{ or } \sigma T^2 + \sigma R^2)}$$

$$Ex.\ \sigma HD_t = \sqrt{((0.003)^2 + 0.002^2 + (1.5 * \tan(20/4''))^2 + 0.002^2)}$$

(Continuing from above, using an instrument centering of 2mm, a pole and bipod at 1.5m HT and a standard Leica circular prism)

$$\sigma HD_t = 0.0047m$$

Error in determining the vertical distance (σVD)

The vertical distance (VD) is the slope distance with the horizontal component removed. It is a function of the SD and the zenith angle (ZA).

$$VD = SD * \cos(ZA)$$

Using error propagation, we can determine the standard deviation of the vertical distance (σVD) is equal to;

$$\sigma VD = \sqrt{((\partial VD/\partial SD)^2 * \sigma SD^2 + (\partial VD/\partial ZA)^2 * \sigma ZA^2)}$$

$$(\partial VD/\partial SD) = \cos(ZA)$$

$$(\partial VD/\partial ZA) = -SD * \sin(ZA)$$

$$\sigma VD = \sqrt{((\cos(ZA)^2 * (\sigma E_c + (\sigma E_s * SD))^2 + (-SD * \sin(ZA))^2 * (\sigma d / \sqrt{n})^2)}$$

$$Ex.\ \sqrt{((\cos(90.5)^2 * (0.002 + ((2/1000000)*500))^2 + (-500 * \sin(90.5))^2 * (3'' * (\pi/648{,}000) / \sqrt{1})^2)}$$

(for a 2mm +/- 2 PPM, 3" instrument @ a SD = 500m and ZA = 90° 30' 00")

$$\sigma VD = 0.0073m$$

Error in determining the elevation of a point (σEL)

The standard deviation of the elevation of the point we are measuring to is the sum of errors of the measured vertical distance (σVD) and the standard deviation of how well we measure the HT (σV_t) and HI (σV_i).

$$\sigma EL = \sqrt{(\sigma VD^2 + \sigma V_t^{\,2} + \sigma V_i^{\,2})}$$

$$Ex.\ \sigma EL = \sqrt{(0.0073^2 + 0.001^{\,2} + 0.003^2)}$$

(Continuing from above, using an instrument vertical centering of 1mm and target vertical centering of 3mm)

$$\sigma EL = +/- 0.0079m$$

$$\sigma EL_{95\%} = 2 * (\sigma EL) = +/- 0.0159m$$

Error in determining the horizontal coordinates of a point (σN + σE)

The standard deviation of the coordinates of the point we measure to are a function of the azimuth (AZ) and the horizontal distance (HD)

$$\Delta N = HD * \cos(AZ)$$

Using error propagation, we can determine the standard deviation of the relative northing of the measured point (σN) is equal to;

$$\sigma N = \sqrt{((\partial N/\partial HD)^2 * \sigma HD_T^2 + (\partial N/\partial AZ)^2 * \sigma AZ^2)}$$

$$\sigma N = \sqrt{(\cos(AZ)^2 * \sigma HD_T^2 + (-HD*\sin(AZ))^2 * \sigma AZ^2)}$$

$$Ex.\ \sigma N = \sqrt{(\cos(155)^2 * (0.0047)^2 + (-499.981*\sin(155))^2 * ((4.49'')^2)}$$

(Continuing from our above example and AZ = 155° 00' 00")

$$\sigma N = 0.0062m$$

$$\sigma N_{95\%} = 2 * (\sigma N) = +/- 0.0124m$$

$$\Delta E = HD * \sin(AZ)$$

Using error propagation, we can determine the standard deviation of the relative easting of the measured point (σE) is equal to;

$$\sigma E = \sqrt{((\partial E/\partial HD)^2 * \sigma HD_T^2 + (\partial E/\partial AZ)^2 * \sigma AZ^2)}$$

$$\sigma E = \sqrt{(\sin(AZ)^2 * \sigma HD_T^2 + (HD*\cos(AZ))^2 * \sigma AZ^2)}$$

$$Ex.\ \sigma E = \sqrt{(\sin(155)^2 * (0.0047)^2 + (-499.981*\cos(155))^2 * ((4.49'')^2)}$$

(Continuing from our above example and AZ = 155° 00' 00")

$$\sigma E = 0.0101m$$

$$\sigma E_{95\%} = 2 * (\sigma E) = +/- 0.0202m$$

$$\sigma Hz\ Pos = \sqrt{\ \sigma N^2 + \sigma E^2}$$

$$\sigma Hz\ Pos = +/- 0.0118m$$

$$\sigma Hz\ Pos_{95\%} = +/- 0.0237m$$

Conclusion

The purpose of these formulae and the accompanying worksheet are to determine two things.

1. Statistically, how precise are you measuring/how precise are the coordinates you are deriving.
2. Given a required precision, what variables can you adjust to meet that precision.

For example, we are asked to "layout some building corners to within +/- 3mm."

This is a common request from a client, and because it isn't very specific, we will have to make a few assumptions.

A. The requested precision will be relative, and we will assume our control is without error. Usually, a project must be relatively more precise than absolutely, as the structures and utilities within the scope of a project need to tie in more precisely with each other than with structures and utilities outside of the project scope. Not to say absolute precision isn't important. But we can assume the control network has considered the absolute accuracy required.
B. It is rare that a confidence level accompanies a tolerance specified by a client, and there is a good chance they won't fully understand what a standard deviation is or how that relates to confidence intervals. Although it will require more stringent controls, we should usually err on the side of safety and use 95% confidence **depending on what the layout is for**. I've found it common that a client will ask for excessively precise tolerances without a full understanding of what is required to achieve them. That being said, for this task I understand these building corners are only for concrete formwork and probably don't actually require 95% confidence of 3mm. There will most likely be deviations in the physical formwork of more than 3mm. We will work towards 1 standard deviation of confidence for this example.
C. We can ignore elevations in this example as they were not asked for.

Continuing with our example above we are currently sitting at a σHz Pos = 12mm, not very close to our required precision. Since we have control setup around our worksite, we can ensure our max SD < 50m. That alone gets our σHz Pos = 5.6mm. Next, we are going to swap out our 1.5m rod/bipod to a mini with a HT of 0.100m which gets us to 4.7 mm. Because we are staking out, it is difficult and time consuming to make multiple sets to each interation of a stake out point, so we will refrain from adjusting our n value. However, taking the instrument off tracking mode got us down to 4.3 mm. At this point it becomes very difficult to increase our precision. If we swap out our instrument for a +/- 1" 1mm +/- 1PPM instrument, we only get down to 4.1mm. To get that last little bit of precision we have two options. We need to either center our instrument to within 1mm (This may be doable with extra care and ensuring the instrument optical plummet is perpendicular to the vertical axis by centering over the control point and rotating the instrument and ensuring there is no deviation) or using a prism with a lower σR. If we keep our mini and take great care in centering the instrument to +/- 1mm we get our a σHz Pos = 3.3mm or rounding down to +/- 3mm.